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Boundary and particle number effects on the thermodynamic properties of trapped ideal Bose gases

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Abstract. The ideal noninteracting Bose gases trapped in a generic power-law potential in an any-dimensional space are studied. We present theoretical results of the corrections of thermodynamic properties due to finite particle number effects. The calculation uses the Euler-Maclaurin approximation to simplify the condensate fraction, and it also uses the *Maslov index* to discuss the boundary effect. Recently BEC (Bose-Einstein Condensation) has also been observed in a microelectronic chip; therefore, with a similar microstructure, we can obtain the effects of a rigid wall in a trap that have never been found before.

PACS. 05.30.-d Quantum statistical mechanics

1 Introduction

Bose-Einstein condensation (BEC) was first predicted by Einstein [1] in 1925 and was observed in a series of remarkable experiments on the vapors of rubidium [2], lithium [3], and sodium [4]. For a better understanding of the BEC behavior theoretically, some important events are worth being mentioned. The theory for a noninteracting ideal Bose gas confined in a rigid container had already been given in a textbook [5]. Then de Groot *et al.* [6] in 1950 calculated the BEC in a harmonic trap and Grossmann *et al.* [7] in 1995 studied the finite-size effects. Also Bagnato *et al.* [8] in 1987 presented their semiclassical calculation results for BEC in a general power-law anisotropic potential trap, with different shapes in different directions.

The noniteracting ideal Bose gas model is easy to calculate and it points out the main features of BEC. Also the measured transition temperature was found to be very close to the value predicted by the ideal Bose gas model. Conventionally there are two kinds of methods to calculate the thermodynamic quantities (especially the condensate temperature and condensate fraction) of ideal Bose gases in a power-law potential trap. The first method considers the classically continuous limit of the energy spectrum of a single particle in the trap and integrates the continuous density of states to get the desired quantities [8]. The advantage of this method is that it can easily explore the shape and dimension effects for various kinds of potentials. But it ignores the finiteness of the energy spectrum spacing and the ground state energy, which can play important roles in determining the thermodynamic properties of a finite system. The second method is only used for the harmonic type traps [9], where we are able to summarize the discrete spectrum with high accuracy for the Bose-Einstein distribution and obtain the first order corrections of the finite-size effects.

To carefully deal with a generic power-law anisotropic potential trap, we develop a general method that combines the advantages of the two above. Our method includes two procedures:

In procedure I: First we find the energy spectrum of a trapped particle, where a semiclassical quantization rule [10] is used which can give very accurate energy spectra for various power-law potentials even when there are sharp boundaries in the space. The influences of the shape of the trapping potential and the sharp boundaries in the space on the energy spectrum of the trapped particle are included in two parameters: the potential power and the Maslov index respectively. The central idea is that both the geometry of the potential (the power) and the topology of the space (the boundary) influence the energy spectrum. We show three cases of different Maslov indices in Figure 1.

In procedure II: We calculate the sum of the Bose-Einstein distribution over all discrete spectra by the Euler–Maclaurin summation formula. The first to use this formula to deal with BEC was Haugset *et al.* [11]; it calculates the thermodynamic properties in an anisotropic harmonic trap for a finite number of particles very accurately. This formula is a discrete summation which can be

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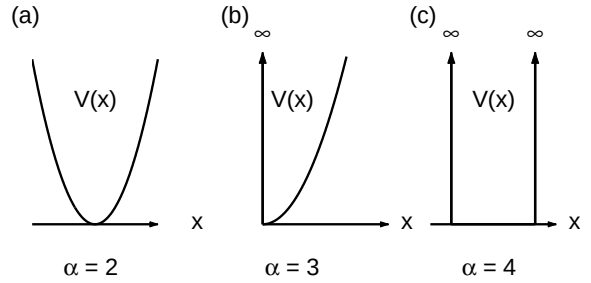

Fig. 1. The Maslov indices α for three cases; they are determined by the shapes of the potentials. In (a), the potential with a finite t has $\alpha = 2$. In (b), the potential is formed by the right half of the potential as in (a), plus a rigid wall located at $x = 0$. In (c), the potential is two infinite walls, the potential power $t = \infty$ and $\alpha = 4$ (see Eq. (1)).

written as an integral plus an infinite series of correction terms. We make good use of the discrete energy spectra from *procedure I*, and it can be extended to calculate various power-law potential problems.

This paper is organized as follows: In Section 2, we give the energy spectrum derived by the semicassical quantization procedure. Then we introduce the Euler–Maclaurin summation formula and show the relation between the original summation and the corresponding integration [12]. In Section 3, we present the zero order result of our method for a generic d-dimension power-law potential trap as reported by others [13]. In Section 4, by considering the first order corrections in an anisotropic harmonic trap, we show that the same finite-size effects as in reference [14] can be deduced. In Section 5, we derive the condensate fraction in a general d-dimensional power-law potential trap and its first order corrections; these results have never been derived before. We also obtain the well-defined transition temperature T_c for a finite number of particles. In Section 6, we discuss an example where the potential is an ordinary harmonic trap and the particles are confined in the upward half-space by a rigid wall below. In Section 7, another example is discussed where the particles are confined in a two-dimensional harmonic trap that is between two rigid walls. Discussions of the results are presented in Section 8.

2 Single particle energy spectrum and Euler-Maclaurin formula

Let us first consider a particle of mass m in a generic power-law potential in a d-dimensional space. The single-particle Hamiltonian is given by:

$$H = \frac{p^2}{2m} + \sum_{i=1}^{d} U_i \left| \frac{x_i}{L_i} \right|^{t_i}, \tag{1}$$

where U_i is the coupling constant, L_i is the acting range, and t_i is the potential power.

To derive the energy spectrum we use the semiclassical quantization rule:

$$\oint p_i \, \mathrm{d}q_i = 2\pi \left(n_i + \frac{\alpha_i}{4} \right) \hbar, \quad i = 1, 2, \ldots, d, \tag{2}$$

where α_i is the Maslov index of the ith dimension.

Applying (2) to the ith dimension we obtain the energy levels:

$$E_{n_i} = A_i \left(n_i + \frac{\alpha_i}{4} \right)^{2t_i/(t_i+2)}, (n_i = 0, 1, 2, 3, \cdots), \tag{3}$$

where

$$A_i = U_i^{2/(t_i+2)} \left(\frac{\pi\hbar^2}{2mL_i^2} \right)^{t_i/(t_i+2)} \left[\frac{\Gamma(\frac{3}{2} + \frac{1}{t_i})}{\Gamma(1 + \frac{1}{t_i})} \right]^{2t_i/(t_i+2)}. \tag{4}$$

A system of N noninteracting bosons can only be treated analytically in the grand-canonical ensemble. The Bose-Einstein distribution function for a bosonic system is given by:

$$n(E) = \frac{1}{\mathrm{e}^{(E-\mu)/kT} - 1} = \sum_{j=1}^{\infty} z^j \mathrm{e}^{-jE/kT}, \tag{5}$$

where $z = \exp(\mu/kT)$ is the fugacity, μ is the chemical potential, and $n(E)$ represents the number of particles in the energy level E, given by:

$$E = \sum_{i=1}^{d} E_{n_i}. \tag{6}$$

The total number of particles is:

$$N = \sum_{\{n\}} n(E) = \sum_{j=1}^{\infty} z^j \prod_{i=1}^{d} \sum_{n_i=0}^{\infty} \mathrm{e}^{-jE_{n_i}/kT}. \tag{7}$$

The sum runs over all possible states. Usually one goes to the classical (continuous) limit to approximate the sum $\sum_{n_i=0}^{\infty} \mathrm{e}^{-jE_{n_i}/kT}$ as an integral:

$$\sum_{i=0}^{n} f(i) \approx \int_0^n f(x)\mathrm{d}x. \tag{8}$$

To improve the accuracy, we employ the Euler-Maclaurin summation formula [12]:

$$\sum_{i=0}^{n} f(i) = \int_0^n f(x)\mathrm{d}x - B_1[f(0) + f(n)] + \sum_{p=1}^{q} \frac{1}{(2p)!} B_{2p} \left[f^{(2p-1)}(n) - f^{(2p-1)}(0) \right] + \text{remainder term}, \tag{9}$$

$$\text{remainder term} = \frac{-1}{(2q)!} \int_0^n f^{(2q)}(x) B_{2q}(x)\mathrm{d}x. \tag{10}$$

Here each B_i is a Bernoulli number. For example, $B_1 = -1/2$, $B_2 = 1/6$, and $B_4 = -1/30$. The $B_{2q}(x)$'s are the Bernoulli functions, and the first two of them are:

$$B_2(x) = x^2 - x + \frac{1}{6},$$
$$B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}\cdot \tag{11}$$

Formula (9) plays the role of a bridge that connects the discrete summation with the continuous integration, and it enables us to find more information than with only the classical limit result.

In this paper we calculate the first order corrections to the classical limit. These corrections include the effects of the finite particle number and the Maslov index. In the following sections we will present the details and discuss two examples.

3 Classical limit

In the classical limit $\hbar \to 0$, the Maslov index α_i can be omitted. We have:

$$\begin{aligned}\sum_{n_i=0}^{\infty} \mathrm{e}^{-jE_{n_i}/kT} &\simeq \int_0^{\infty} \mathrm{e}^{-jE_{n_i}/kT}\mathrm{d}n_i,\\ &= \left(\frac{kT}{jA_i}\right)^{(1/2+1/t_i)} \Gamma\left(\frac{3}{2}+\frac{1}{t_i}\right),\\ &= \left(\frac{kT}{jA_i}\right)^{\eta_i} \Gamma(\eta_i+1). \end{aligned}\tag{12}$$

For simplicity, we define the variables:

$$\eta_i \equiv \frac{1}{2} + \frac{1}{t_i}, \tag{13}$$
$$\eta \equiv \sum_{i=1}^{d} \eta_i, \tag{14}$$
$$\begin{aligned} A &\equiv \prod_{i=1}^{d} A_i^{-\eta_i} \Gamma(1+\eta_i),\\ &= \frac{(2m)^{d/2}}{\hbar^d \pi^{d/2}} \prod_{i=1}^{d} \frac{L_i \Gamma(1+\frac{1}{t_i})}{U_i^{1/t_i}}\cdot \end{aligned}\tag{15}$$

The total particle number now can be written as:

$$\begin{aligned} N &= \sum_{j=1}^{\infty} z^j \prod_i \sum_{n_i=0}^{\infty} \mathrm{e}^{-jE_{n_i}/kT},\\ &= A(kT)^{\eta} \sum_{j=1}^{\infty} \frac{z^j}{j^{\eta}},\\ &\equiv A(kT)^{\eta} g_{\eta}(z), \end{aligned}\tag{16}$$

where $g_\eta(z)$ is the familiar Bose-Einstein function. It is sometimes written as $\mathrm{Li}_\eta(z)$.

The transition temperature for the classical limit corresponding to $z \to 1$ is given by:

$$T_c^0 = \frac{1}{k}\left(\frac{N}{A\zeta(\eta)}\right)^{1/\eta}, \tag{17}$$

where $\zeta(\eta)$ is the Riemann zeta function and we have used the relation:

$$g_\eta(1) = \zeta(\eta). \tag{18}$$

The condensate fraction N_0/N can also be obtained as:

$$\frac{N_0}{N} = 1 - \frac{N_e}{N} = 1 - \left(\frac{T}{T_c}\right)^{\eta}, \tag{19}$$

where N_0 and N_e represent the number of particles in the ground state and the other excited states, respectively. Obviously, this result is the same as in reference [8], because we derive this result under the same classical (continuous) limit.

4 Harmonic trap

In this section we use our new method to handle the d-dimensional anisotropic harmonic-oscillator trap. In this trap, we have $t_i = 2$, $\alpha_i = 2$ for each i-dimension, and the energy level is:

$$E_{n_i} = \left(n_i + \frac{1}{2}\right)\hbar\omega_i. \tag{20}$$

The ground state energy for a single particle is:

$$E_0 = \sum_{i=1}^{d} \frac{\hbar\omega_i}{2} = \frac{d\hbar\bar{\omega}}{2}\cdot \tag{21}$$

Hereafter we define two variables $\bar{\omega}$ and Ω as:

$$\bar{\omega} \equiv \frac{1}{d}\sum_{i=1}^{d} \omega_i, \tag{22}$$
$$\Omega \equiv \left(\prod_{i=1}^{d} \omega_i\right)^{1/d}. \tag{23}$$

Note that below the transition temperature T_c ,the chemical potential μ approaches ground state energy, not zero. We further define two variables as:

$$\epsilon \equiv E_0 - \mu, \tag{24}$$
$$z' \equiv \exp\left(-\frac{\epsilon}{kT}\right), \tag{25}$$

then the ground state population can be written as:

$$N_0 = \frac{1}{\mathrm{e}^{(E_0-\mu)/kT} - 1} = \frac{z'}{1-z'}\cdot \tag{26}$$

Note that near T_c, it is z' that approaches 1, not z.

Now we use equation (9) to calculate $\sum_{n_i} \mathrm{e}^{-jE_{n_i}/kT}$ and keep only the first order corrections:

$$\sum_{n_i=0}^{\infty} \mathrm{e}^{-jE_{n_i}/kT} \simeq \int_0^{\infty} \mathrm{e}^{-j\left[(n_i+\frac{1}{2})\hbar\omega_i\right]/kT} \mathrm{d}n_i + \frac{\mathrm{e}^{-j\hbar\omega_i/2kT}}{2}, \approx \frac{kT}{j\hbar\omega_i}\left(1 - \frac{j\hbar\omega_i}{2kT}\right) + \frac{1}{2} = \frac{kT}{j\hbar\omega_i}. \quad (27)$$

Here we have assumed $j\hbar\omega_i \ll kT$, which is correct except for $T \ll T_c$. It is interesting to note that in the above equation, the factor 1/2 provided by the ground state energy is equal in magnitude and opposite in sign to the Bernoulli number B_1, and that they cancel each other to make the first order correction disappear. From this result we get the total particle number:

$$N = \sum_{j=1}^{\infty} z^j \prod_{i=1}^{d} \sum_{n_i=0}^{\infty} \mathrm{e}^{-jE_{n_i}/kT}, = \left(\frac{kT}{\hbar\Omega}\right)^d \sum_{j=1}^{\infty} \frac{z^j}{j^d}, \quad (28)$$

which can also be written as:

$$N = \left(\frac{kT}{\hbar\Omega}\right)^d \sum_{j=1}^{\infty} \frac{z'^j}{j^d} \mathrm{e}^{jE_0/kT}, \simeq \left(\frac{kT}{\hbar\Omega}\right)^d \sum_{j=1}^{\infty} \frac{z'^j}{j^d}\left(1 + \frac{jE_0}{kT}\right), = \left(\frac{kT}{\hbar\Omega}\right)^d g_d(z') + \frac{E_0}{\hbar\Omega}\left(\frac{kT}{\hbar\Omega}\right)^{d-1} g_{d-1}(z'). \quad (29)$$

In classical limit $\hbar \to 0$ then $E_0 \to 0$, the transition temperature is given by:

$$T_c^0 = \frac{\hbar\Omega}{k}\left[\frac{N}{g_d(1)}\right]^{1/d}. \quad (30)$$

Here we used $\hbar N^{1/d}$ which is a finite number. It means that the system is in the thermodynamic limit $N \to \infty$. Considering the rightside of the second term of equation (29), the well-defined transition temperature for the finite system is:

$$\frac{T_c}{T_c^0} = 1 - \frac{g_{d-1}(1)}{d[g_d(1)]^{1-1/d}} \frac{E_0}{\hbar\Omega} N^{-1/d}. \quad (31)$$

The correction term is a consequence of $E_0 \neq 0$ and $N \neq \infty$.

We also have the occupation fraction for the excited states:

$$\frac{N_e}{N} = \left(\frac{T}{T_c^0}\right)^d + \frac{E_0}{kT_c^0}\frac{g_{d-1}(1)}{g_d(1)}\left(\frac{T}{T_c^0}\right)^{d-1} \quad (32)$$

and the condensate fraction for the ground state at temperatures $T < T_c$:

$$\frac{N_o}{N} = 1 - \left(\frac{T}{T_c^0}\right)^d - \frac{E_0}{kT_c^0}\frac{g_{d-1}(1)}{g_d(1)}\left(\frac{T}{T_c^0}\right)^{d-1}. \quad (33)$$

Our results are the same as the previous work [13]. Thus our method is as accurate as the previous method in a harmonic trap and we can extend it to a general power-law trap.

5 General power-law trap

In a general power-law trap, the Maslov index α_i may not be equal to 2, so we have the summation approximation:

$$\sum_{n_i=0}^{\infty} \mathrm{e}^{-jE_{n_i}/kT} \simeq \left(\frac{kT}{jA_i}\right)^{\eta_i} \Gamma(\eta_i+1) - \left(\frac{\alpha_i - 2}{4}\right), \simeq \left(\frac{kT}{jA_i}\right)^{\eta_i} \Gamma(\eta_i+1)\exp\left[\beta_i\left(\frac{j}{kT}\right)^{\eta_i}\right], \quad (34)$$

where

$$\beta_i \equiv \frac{(\alpha_i - 2)}{4\Gamma(1+\eta_i)} A_i^{\eta_i}. \quad (35)$$

The particle number now can be expressed as:

$$N = \sum_{j=1}^{\infty} z^j \prod_i \sum_{n_i=0}^{\infty} \mathrm{e}^{-jE_{n_i}/kT}, = (kT)^\eta A \sum_{j=1}^{\infty} \frac{z^j}{j^\eta} \prod_i \exp\left(-\beta_i (j/kT)^{\eta_i}\right), \simeq A(kT)^\eta g_\eta(z') + E_0 A (kT)^{\eta-1} g_{\eta-1}(z') - A\sum_{i=1}^{d} (kT)^{\eta-\eta_i} \beta_i g_{\eta-\eta_i}(z'). \quad (36)$$

This expression has one main term and two correction terms. The domain of function $g_n(1)$ is $n > 1$, thus the correction terms can neither be used in the 3-D rigid container case, nor in the 2-D harmonic case. Hereafter we just discuss the situations with the condition that all the function is in domain, which is applicable to most traps.

The well-defined transition temperature T_c for the finite system is constrained by:

$$N = A(kT_c)^\eta g_\eta(1) + E_0 A (kT_c)^{\eta-1} g_{\eta-1}(1) - A\sum_{i=1}^{d} (kT_c)^{\eta-\eta_i} \beta_i g_{\eta-\eta_i}(1), \quad (37)$$

which is a little different from its thermodynamic-limit counterpart T_c^0. Denote the ratio between these two characteristic temperatures as:

$$\frac{T_c}{T_c^0} \simeq 1 - \Delta t_c, \quad (38)$$

then the shift:

$$\Delta t_c \equiv \frac{E_0 g_{\eta-1}(1)}{\eta k T_c^0 g_\eta(1)} - \sum_{i=1}^{d} \frac{\beta_i g_{\eta-\eta_i}(1)}{\eta (kT_c^0)^{\eta_i} g_\eta(1)} \tag{39}$$

represents the finite size correction to T_c^0.

The first correction term is the ordinary finite-size effect; it is caused by the nonzero ground energy level E_0 and is proportional to $N^{-1/\eta}$. The second terms are new ones; they appear only when $\alpha_i \neq 2$ and are proportional to $N^{-\eta_i/\eta}$ for each i. Thus for a smooth potential like that in Figure 1a this term vanishes.

The occupation fraction in excited states at temperatures $T < T_c$ can be derived from equation (37) as:

$$\frac{N_e}{N} \simeq \left(\frac{T}{T_c^0}\right)^\eta + \frac{g_{\eta-1}(1)}{g_\eta}\frac{E_0}{kT_c^0}\left(\frac{T}{T_c^0}\right)^{\eta-1} - \sum_{i=1}^{d} \frac{g_{\eta-\eta_i}(1)}{g_\eta(1)} \frac{\beta_i}{(kT_c^0)^{\eta_i}} \left(\frac{T}{T_c^0}\right)^{\eta-\eta_i}. \tag{40}$$

Equation (40) indicates that the occupation fraction of the excited states is more important when the system has a finite number of particles. We can also obtain the condensate fraction for $T < T_c$:

$$\frac{N_0}{N} = 1 - \frac{N_e}{N} \simeq 1 - \left(\frac{T}{T_c^0}\right)^\eta - \frac{E_0 g_{\eta-1}(1)}{kT_c^0 g_\eta}\left(\frac{T}{T_c^0}\right)^{\eta-1} + \sum_{i=1}^{d} \frac{\beta_i g_{\eta-\eta_i}(1)}{(kT_c^0)^{\eta_i} g_\eta(1)} \left(\frac{T}{T_c^0}\right)^{\eta-\eta_i}. \tag{41}$$

Here the finiteness of N always makes this fraction decrease.

In the following two sections we will apply these formulas to study two concrete examples.

6 Example 1

In the first example we consider the trapping potential:

$$U(z) = \begin{cases} \frac{1}{2}m\omega^2 r^2, & z > 0 \\ \infty, & z < 0 \end{cases}, \tag{42}$$

then we have:

$$\alpha_1 = \alpha_2 = 2, \quad \alpha_3 = 3, \tag{43}$$

$$t_1 = t_2 = 2, \quad t_3 = 2 \ \text{(oneside)}. \tag{44}$$

Remember that the trapping potential along the 3rd dimension consists of one half of the original harmonic type potential, together with a rigid wall. Thus, according to the rules and formulas discussed in Section 2, we get:

$$\eta_1 = \eta_2 = \eta_3 = 1, \quad \eta = 3,$$
$$A_1 = A_2 = \hbar\omega, \quad A_3 = 2\hbar\omega, \quad A = \frac{1}{2(\hbar\omega)^3},$$
$$E_{1,0} = E_{2,0} = \frac{\hbar\omega}{2}, \quad E_{3,0} = \frac{3\hbar\omega}{2}, \quad E_0 = \frac{5\hbar\omega}{2} \tag{45}$$

and:

$$\beta_1 = 0, \quad \beta_2 = 0, \quad \beta_3 = \frac{\hbar\omega}{2}. \tag{46}$$

The total particle number is given by:

$$N = \frac{1}{2}\left(\frac{kT}{\hbar\omega}\right)^3 g_3(z') + \left(\frac{kT}{\hbar\omega}\right)^2 g_2(z'), \tag{47}$$

and the transition temperature in the thermodynamic limit is:

$$T_c^0 = \frac{\hbar\omega}{k}\left[\frac{2N}{g_3(1)}\right]^{1/3}. \tag{48}$$

This result is $2^{1/3}$ times larger than the result obtained in the case without rigid walls. The transition temperature shift ratio is:

$$\Delta t_c = \frac{2\hbar\omega g_2(1)}{3kT_c^0 g_3(1)} = \frac{2g_2(1)}{3g_3(1)}\left(\frac{g_3(1)}{2N}\right)^{1/3}, \tag{49}$$

which is $4/3 \times 2^{1/3}$ times larger than the value obtained in the no-rigid wall case. Finally, we obtain the condensate fraction:

$$\frac{N_0}{N} = 1 - \left(\frac{T}{T_c^0}\right)^3 - \frac{2\hbar\omega g_2(1)}{kT_c^0 g_3(1)}\left(\frac{T}{T_c^0}\right)^2 \tag{50}$$

in the $T < T_c$ region. The normalized finite-size effect term is $4/3 \times 2^{-1/3}$ times larger than in the no-rigid wall case.

Some numerical results of this example are shown in Figure 2.

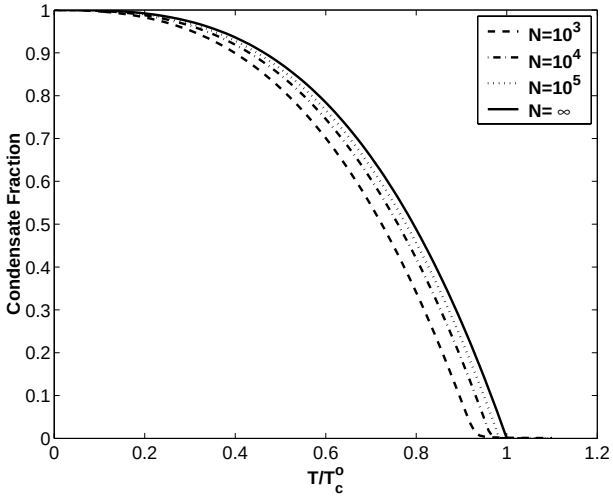

Fig. 2. Numerical results for example 1. The condensate fraction as a function of temperature for $N = 10^3, 10^4, 10^5$, and ∞.

7 Example 2

Now we consider the potential:

$$U(z) = \begin{cases} \frac{1}{2}m\omega^2(x^2+y^2), & 0 < z < L \\ \infty, & z < 0 \text{ or } z > L \end{cases}. \quad (51)$$

This potential in the third dimension (the z direction) is an infinite well potential centered at $z = L/2$ and has a half-width $L_3 = L/2$. Thus we have:

$$\alpha_1 = \alpha_2 = 2, \quad \alpha_3 = 4, \quad (52)$$

and

$$t_1 = t_2 = 2, \quad t_3 = \infty. \quad (53)$$

The quantization rule in Section 2 gives us:

$$\eta_1 = \eta_2 = 1, \quad \eta_3 = \frac{1}{2}, \quad \eta = \frac{5}{2},$$
$$A_1 = A_2 = \hbar\omega, \ A_3 = \delta\hbar\omega, \ A = \frac{\sqrt{\pi}}{2\sqrt{\delta}(\hbar\omega)^{5/2}},$$
$$E_{1,0} = E_{2,0} = \frac{\hbar\omega}{2}, \ E_{3,0} = \delta\hbar\omega, \ E_0 = (1+\delta)\hbar\omega, \quad (54)$$

where m is the mass of the particle. The parameter δ is defined by:

$$\delta \equiv \frac{\hbar\pi^2}{2mL^2\omega}, \quad (55)$$

and the β's are given by:

$$\beta_1 = 0, \quad \beta_2 = 0, \quad \beta_3 = \sqrt{\frac{\delta\hbar\omega}{\pi}}. \quad (56)$$

The total particle number now becomes:

$$N = \frac{1}{2}\sqrt{\frac{\pi}{\delta}}\left(\frac{kT}{\hbar\omega}\right)^{5/2} g_{5/2}(z') + \frac{1+\delta}{2}\sqrt{\frac{\pi}{\delta}}\left(\frac{kT}{\hbar\omega}\right)^{3/2} g_{3/2}(z') - \frac{1}{2}\left(\frac{kT}{\hbar\omega}\right)^{2} g_2(z'), \quad (57)$$

and the transition temperature in the thermodynamic limit is:

$$T_c^0 = \frac{\hbar\omega}{k}\left(\frac{\delta}{\pi}\right)^{1/5}\left(\frac{2N}{g_{5/2}(1)}\right)^{2/5}, \quad (58)$$

with the temperature shift ratio:

$$\Delta t_c = \frac{2(1+\delta)g_{3/2}(1)}{5g_{5/2}(1)}\frac{\hbar\omega}{kT_c^0} - \frac{2g_2(1)}{5g_{5/2}(1)}\sqrt{\frac{\delta\hbar\omega}{\pi kT_c^0}} = \frac{2(1+\delta)g_{3/2}(1)}{5g_{5/2}(1)}\left(\frac{\pi}{\delta}\right)^{1/5}\left(\frac{g_{5/2}(1)}{2N}\right)^{2/5} - \frac{2g_2(1)}{5g_{5/2}(1)}\left(\frac{\delta}{\pi}\right)^{2/5}\left(\frac{g_{5/2}(1)}{2N}\right)^{1/5}. \quad (59)$$

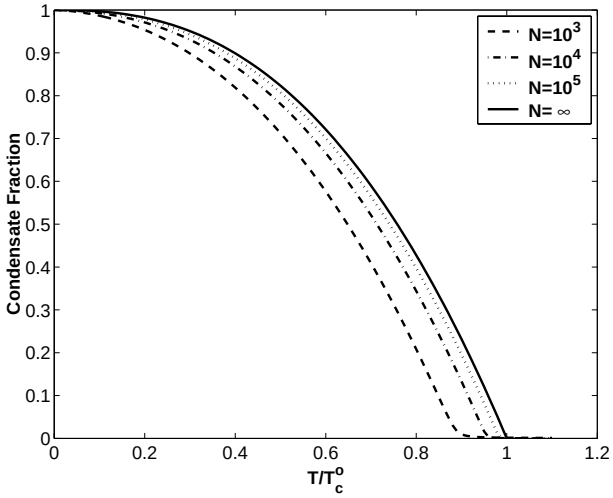

Fig. 3. Numerical results for example 2. The condensate fraction as a function of temperature for $N = 10^3, 10^4, 10^5$, and ∞ at $\delta = 0.01$.

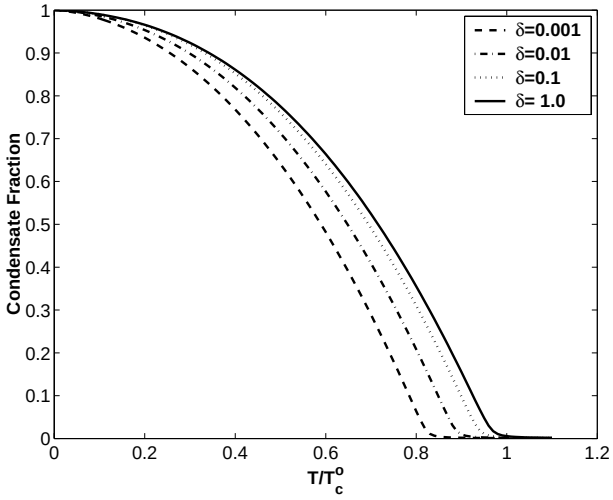

Fig. 4. Numerical results for example 2. The condensate fraction as a function of temperature for $\delta = 0.001, 0.01, 0.1$, and 1.0 at $N = 1000$.

Finally we obtain the condensate fraction:

$$\frac{N_0}{N} = 1 - \left(\frac{T}{T_c^0}\right)^{5/2} - \frac{(1+\delta)g_{3/2}(1)}{g_{5/2}(1)}\frac{\hbar\omega}{kT_c^0}\left(\frac{T}{T_c^0}\right)^{3/2} + \frac{g_2(1)}{g_{5/2}(1)}\sqrt{\frac{\delta\hbar\omega}{\pi kT_c^0}}\left(\frac{T}{T_c^0}\right)^{2} \quad (60)$$

in the $T < T_c$ region.

Numerical results for this example are shown in Figures 3 and 4.

8 Discussion

We use the grand canonical ensemble to study the thermodynamic properties of a noninteracting boson gas that

is trapped in a power-law potential. Since in this noninteracting case the total energy of the whole system can simply be written as the sum of the energy of each individual particle, the original many body problem is reduced to a one particle problem. Moreover, since the kinetic energy and the trapping potential discussed in this paper are separable in each dimension, the d-dimensional problem is also reduced to a one dimensional problem. Thus this method is able to deal with various power-law potentials and dimensionality problems, something not achieved by other methods. Accordingly, we show more clearly that the energy spectrum of each dimension is independent, and it is easy to understand the role each dimension plays in the BEC. We then apply the semiclassical quantization rule to each dimension of the power-law potential trap to get the energy spectrum of the system. Finally, we use the Euler-Maclaurin summation formula to sum over the Bose-Einstein distribution and get the desired quantities such as the transition temperature and condensate fraction for a finite number particles. This is what we did in this paper.

The drawback of this method is that the divergence mentioned enters the mathematics. We have not yet found a general method to treat it. To avoid the divergence, Haugset *et al.* [11] used the Euler–Maclaurin approximation by another parameter expansion in a harmonic trap. They were able to derive higher-order corrections cleverly but it is restricted to a power $t_i = 2$. In the 3-D rigid container case, the correction also encounters the same difficulty. Pathria applied another technique called the Poisson summation formula to get the correction [15], but this technique is hard to extend to any other case.

Recently, BEC has also been observed in a microelectronic chip [16]. In this microstructure, one needs to consider the effect caused by β_i. In the future, we expect that there will be more BEC phenomena observed in microsystems, and our work may provide some useful insights into the understanding of those systems.

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